Extended Robust Model Predictive Control

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DOI 10.1002/aic.10175
Published online in Wiley InterScience (www.interscience.wiley.com).

The model predictive control (MPC) of stable linear systems with model uncertainty is addressed. A strategy is proposed to extend existing robust controllers that were developed for the regulator problem, to the general case in which the system equilibrium point is unknown. Nominal stability is obtained with the consideration of an infinite output horizon. The method is based on a state-space model formulation that allows a straightforward solution to the Lyapunov equation. The method is detailed for two particular controllers of the robust MPC literature. Simulation examples illustrate the performance of the proposed approach and demonstrate that it can be implemented in real applications. © 2004 American Institute of Chemical Engineers *AIChE J,* 50: 1824–1836, 2004
Keywords: predictive control, robust stability, receding horizon, model uncertainty

Introduction

Model predictive control (MPC) is a control strategy in which the control action is obtained by minimizing, at each sample step, an open-loop cost function subject to constraints on the input amplitude and input moves. In the usual approach, the controller minimizes the distance between the predicted output trajectory and the desired output reference. The cost also includes a term that penalizes the input moves. When the system equilibrium point is not exactly known, the output trajectory can be computed using a discrete time model in the incremental form for the input as follows

$$x(k+1) = Ax(k) + B\Delta u(k) \tag{1}$$

$$y(k) = Cx(k) \tag{2}$$

where $x \in \mathbb{C}^{nx}$ is the vector of states, $u \in \mathbb{R}^{nu}$ is the vector of inputs, k is the present time step, $\Delta u(k) = u(k) - u(k-1)$, and $y \in \mathbb{R}^{ny}$ is the vector of outputs. A, B, and C are matrices with appropriate dimensions. The incremental form of Eq. 1 is suitable for systems with unmeasured disturbances or gain uncertainty. However, it brings some inconveniences when the scope is to design a robust controller. This is so because the incremental form introduces artificial integrating poles into the system model. The consequence is that the resulting model is no longer stable and many important results concerning the robust control of stable systems cannot be directly applied. The main scope of this article is to reformulate the MPC optimization problem such that some existing robust MPC results can be adapted to the model formulation described in Eqs. 1 and 2.

For the nominal model, which usually corresponds to the most frequent operating point of the system, stability can be obtained by several methods. One of these methods imposes a constraint that limits the terminal state to a point or a region with bounded amplitude (Keerthi and Gilbert, 1988; Meadows et al., 1995; Polak and Yang, 1993). This method may result in an infeasible controller if constraints on the input increments are also included into the control optimization problem. Another weakness of this strategy is that, in the case of model uncertainty, it cannot be extended to the robust control problem because the terminal state constraint cannot be simultaneously satisfied by all possible process models. A popular approach to obtain a stable MPC consists in adopting an infinite prediction horizon (Rawlings and Muske, 1993). For stable systems, the infinite-horizon open-loop objective function is reduced to a finite-horizon objective by defining a terminal state penalty, which is obtained from the solution of a Lyapunov equation.

Robust stability is still one of the main weaknesses of the available MPC technology (Qin and Badgwell, 2003). However, this subject has been extensively treated in the control literature (Kothare et al., 1996; Lee and Yu, 1997; Mayne et al., 2000; Morari and Lee, 1999; Ralhan and Badgwell, 2000;

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Vuthandam et al., 1995). A robust controller is meant to guarantee closed-loop stability for different process operating conditions. It is well known that numerous chemical processes are nonlinear but can be approximated by a set of linear models, where each linear model represents the process locally, around a specific operating condition. If the controller is based on a single linear model, it is desirable to ensure that this controller will remain stable for the whole family of models that represent the process. The infinite-horizon MPC has been used as a framework for several categories of robust controllers:

(1) Cost constraining: Badgwell (1997), Ralhan and Badgwell (2000)

(2) State contracting: Kothare et al. (1996), Lu and Arkun (2000)

(3) Minimum worst case controller cost: Lee and You (1997), Scokaert and Mayne (1998), Lee and Cooley (2000)

All these approaches were developed under the assumption that the system is stable and the system model is such that a null input produces a system output that converges asymptotically to the origin. Consequently, it is implicitly assumed that the control objective is bounded for the null input. This assumption is not valid when the system is represented in the state-space model form described in Eqs. 1 and 2, given that this model contains integrating poles. In this work, we focus on the extension of controllers of categories 1 and 3 to the case where the system is represented by models in the incremental form defined in Eqs. 1 and 2.

The infinite-horizon MPC was extended by Badgwell (1997) to the robust control of processes whose model uncertainty is represented by multiple plants. He includes, in the MPC optimization problem, constraints that prevent the cost associated with the true plant from increasing. Following the same approach, Ralhan and Badgwell (2000) extended the method to the case of stable systems with continuous uncertainty. Kothare et al. (1996) proposed a robust MPC based on the optimal linear state feedback gain, which is obtained as the solution of a min–max optimization problem that is translated into an LMI problem. The approach of Kothare et al. (1996) was extended by Lu and Arkun (2000) to time-varying linear systems with polytopic uncertainty and for the scheduling MPC.

The robust controller based on the min–max solution of the control optimization problem of the infinite-horizon MPC was presented by Lee and Yu (1997). They showed that in this case, the control objective is a Lyapunov function to the dynamic system when the objective is computed considering all possible combinations of system models along the control horizon. A possible disadvantage of this approach is that it can be very conservative, particularly for linear time-invariant systems. The computer time for the solution of the min–max optimization problem may become prohibitive for moderate control horizons. Lee and Cooley (2000) extended the infinite-horizon MPC to the robust regulatory control of systems with time-varying input matrices. They also extended the approach to systems with integrating poles as defined in Eq. 1 in a rather particular way. They defined a primary control objective, which is related to the integrating states of the system. The result of the minimization of this objective is transferred to the min–max control optimization problem as an equality constraint, which guarantees the convergence of the system output to the reference.

The infinite-horizon MPC was further extended to the reference tracking problem by Rodrigues and Odloak (2003). A suitable state-space model of the form defined in Eqs. 1 and 2 was used to obtain the analytical integral of the squared output error along the infinite prediction horizon. One of the advantages of the method is that it is not necessary to compute a terminal state weight through the solution of the discrete Lyapunov equation. The disadvantage of that approach is a more complex formulation of the control optimization problem, which may be difficult to standardize. The method presented here combines the model formulation presented by Rodrigues and Odloak (2003) with the method of Rawlings and Muske (1993) to extend existing robust MPC methods to stable systems with unknown steady state. The model formulation allows an inexpensive solution to the Lyapunov equation to compute the terminal state penalty. The method proposed here follows closely the method of Rodrigues and Odloak (2003), but has a simpler formulation and can be easier to implement in practice. The control objective is modified by including slack variables, such that the infinite-horizon cost remains bounded for the disturbed system when constraints on the inputs become active or model uncertainty is present. With this approach, it is proved that the robust controller drives the true plant output to the desired reference value.

The article is organized as follows. In the next section, we summarize the infinite-horizon control problem with the state-space model in the incremental form. For the particular model realization considered in the control problem, the solution of the Lyapunov equation is detailed. A version of the infinite-horizon MPC is also presented, which has an expanded set of feasible solutions. Then, the approach is applied to extend two existing MPC controllers, which are robust to model uncertainty when operating as regulators. The application of the method is illustrated with simulation examples, followed by appropriate conclusions arising from the study.

The Extended Nominally Stable MPC

Rodrigues and Odloak (2003) presented a state-space model form for stable systems in which the output prediction can be treated as a continuous function of time. This model has its origin in the analytical form of the step response that can be associated with a transfer-function model. The coefficients of this analytical function give rise to the realization of states of the proposed state-space model. Cano and Odloak (2003) extended this particular modeling approach to integrating systems. In this section, we develop the infinite-horizon MPC for models in the incremental form as in Eqs. 1 and 2, following the method of Rodrigues and Odloak (2003), but considering that the output is predicted at discrete sampling instants.

We assume the MIMO (multiple inputs/multiple outputs) stable system with nu inputs and ny outputs, where the Laplace transfer function relating input u_j to output y_i is

$$G_{i,j}(s) = \frac{b_{i,j,0} + b_{i,j,1}s + \cdots + b_{i,j,nb}s^{nb}}{1 + a_{i,j,1}s + \cdots + a_{i,j,na}s^{na}}$$

where $\{na, nb \in \mathbb{N} \mid nb < na\}$. The corresponding step response at time step k can be calculated, for a sampling period T, by the following expression:

$$S_{i,j}(k) = d^0_{i,j} + \sum_{l=1}^{na} [d^d_{i,j,l}] e^{r_{i,j,l}kT}$$

where coefficients $d^0_{i,j}, d^d_{i,j,1}, \ldots, d^d_{i,j,na}$ are obtained by partial fractions expansion and $r_{i,j}, r_{i,j,1}, \ldots, r_{i,j,na}$ are the poles, which are assumed to be distinct. For the sequel of this work, it is convenient to define the following coefficient matrices:

$$D^0 \triangleq \begin{bmatrix} d^0_{1,1} & \cdots & d^0_{1,nu} \\ \vdots & \ddots & \vdots \\ d^0_{ny,1} & \cdots & d^0_{ny,nu} \end{bmatrix} \qquad D^0 \in \mathbb{R}^{ny\times nu}$$

$$D^d \triangleq \text{diag}(d^d_{1,1,1} \cdots d^d_{1,1,na} \cdots d^d_{1,nu,1} \cdots d^d_{1,nu,na} \cdots d^d_{ny,1,1} \cdots d^d_{ny,1,n1} \cdots d^d_{ny,nu,1} \cdots d^d_{ny,nu,na})$$

$$D^d \in \mathbb{C}^{nd\times nd}$$

For this system, a discrete time state-space model, of the form represented in Eqs. 1 and 2, can be written as follows

$$\begin{bmatrix} x^s(k+1) \\ x^d(k+1) \end{bmatrix} = \begin{bmatrix} I_{ny} & 0 \\ 0 & F \end{bmatrix} \begin{bmatrix} x^s(k) \\ x^d(k) \end{bmatrix} + \begin{bmatrix} D^0 \\ D^d F N \end{bmatrix} \Delta u(k) \quad (3)$$

$$y(k) = [I_{ny} \quad \Psi] \begin{bmatrix} x^s(k) \\ x^d(k) \end{bmatrix} \quad (4)$$

where

$$x^s \triangleq [x_1 \quad \cdots \quad x_{ny}]^T \qquad x^s \in \mathbb{R}^{ny}$$

$$x^d \triangleq [x_{ny+1} \quad x_{ny+2} \quad \cdots \quad x_{ny(nuna+1)}]^T$$

$$x^d \in \mathbb{C}^{nd} \qquad nd = nunany$$

$$F \triangleq \text{diag}(e^{r_{1,1,1}T} \cdots e^{r_{1,1,na}T} \cdots e^{r_{1,nu,1}T} \cdots e^{r_{1,nu,na}T} \cdots e^{r_{ny,1,1}T} \cdots e^{r_{ny,1,na}T} \cdots e^{r_{ny,nu,1}T} \cdots e^{r_{ny,nu,na}T})$$

$$F \in \mathbb{C}^{nd\times nd}$$

$$N \triangleq \begin{bmatrix} J_1 \\ \cdots \\ J_{ny} \end{bmatrix} \qquad N \in \mathbb{R}^{nd\times nu} \qquad J_i \triangleq \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$J_i \in \mathbb{R}^{nuna\times nu}$$

$$\Psi \triangleq \begin{bmatrix} \Phi & & 0 \\ & \ddots & \\ 0 & & \Phi \end{bmatrix} \qquad \Psi \in \mathbb{R}^{ny\times nd} \qquad \Phi \triangleq [1 \quad \cdots \quad 1]$$

$$\Phi \in \mathbb{R}^{nuna}$$

The extended infinite-horizon MPC is based on the following cost or control objective

$$V_k \triangleq \sum_{j=0}^{\infty} [e(k+j) - \delta_k]^T Q [e(k+j) - \delta_k] + \sum_{j=0}^{m-1} \Delta u(k+j)^T R \Delta u(k+j) + \delta_k^T S \delta_k \quad (5)$$

where $e(k+j) = y(k+j) - r$ is the output prediction, taking into account the effects of the future control actions; r is the output reference; and $\delta_k \in \mathbb{R}^{ny}$ is a vector of slack variables. $Q \in \mathbb{R}^{ny\times ny}$, $R \in \mathbb{R}^{nu\times nu}$, and $S \in \mathbb{R}^{ny\times ny}$ are assumed positive definite. The slack vector is included in the cost function to prevent the cost from becoming unbounded in the presence of offset in one or more of the controlled outputs. For this purpose, each slack variable refers to the predicted offset in the corresponding controlled output. Weight matrix S should be selected such that the controller tends to reduce the slacks to zero or at least minimize them, depending on the number of inputs, which are not constrained.

The control objective defined in Eq. 5 can be written as follows

$$V_k = \sum_{j=0}^{m} [e(k+j) - \delta_k]^T Q [e(k+j) - \delta_k] + \sum_{j=m+1}^{\infty} [Cx(k+j) - \delta_k - r]^T Q [Cx(k+j) - \delta_k - r] + \sum_{j=0}^{m-1} \Delta u(k+j)^T R \Delta u(k+j) + \delta_k^T S \delta_k \quad (6)$$

With the proposed model formulation, the second term on the right-hand side of Eq. 6 can be developed as follows

$$\sum_{j=m+1}^{\infty} [Cx(k+j) - \delta_k - r]^T Q [Cx(k+j) - \delta_k - r] = \sum_{j=1}^{\infty} [x^s(k+m) - \delta_k - r + \Psi F^j x^d(k+m)]^T \times Q[x^s(k+m) - \delta_k - r + \Psi F^j x^d(k+m)] \quad (7)$$

In this infinite sum, the term $xs(k+m) - \delta_k - r$ does not depend on index j and, therefore, must be made equal to zero to keep the control objective bounded. Thus, the following constraint has to be included in the control optimization problem

$$x^s(k+m) - \delta_k - r = 0 \quad (8)$$

With this condition, Eq. 7 can be simplified as follows

$$\sum_{j=m+1}^{\infty} [Cx(k+j) - \delta_k - r]^T Q [Cx(k+j) - \delta_k - r]$$

$$= \sum_{j=1}^{\infty} x^d(k+m)^T (F^j)^T \Psi^T Q \Psi F^j x^d(k+m) \quad (9)$$

For stable systems, $\lim_{j\to\infty} F^j = 0$, and the infinite sum in the right-hand side of Eq. 9 can be reduced to a single term

$$\sum_{j=1}^{\infty} x^d(k+m)^T (F^j)^T \Psi^T Q \Psi F^j x^d(k+m) = x^d(k+m)^T \bar{Q} x^d(k+m)$$

where $\bar{Q} \in \mathbb{C}^{nd\times nd}$ is such that

$$\bar{Q} - F^T \bar{Q} F = F^T \Psi^T Q \Psi F \quad (10)$$

Equation 10 is the Lyapunov equation for the discrete time system represented in Eqs. 3 and 4. Hence, the control objective defined in Eq. 5 becomes

$$V_k = \sum_{j=0}^{m} [e(k+j) - \delta_k]^T Q [e(k+j) - \delta_k] + x^d(k+m)^T \bar{Q} x^d(k+m) + \sum_{j=0}^{m-1} \Delta u(k+j)^T R \Delta u(k+j) + \delta_k^T S \delta_k \quad (11)$$

To obtain a simpler expression for the above control objective, Eqs. 3 and 4 can be used to evaluate the state and output at future sampling instants as follows

$$y(k+j) = x^s(k+j) + \Psi x^d(k+j) \quad (12)$$

$$x^s(k+j) = x^s(k) + D^0 \Delta u(k) + \cdots + D^0 \Delta u(k+j-1) \quad (13)$$

$$x^d(k+j) = F^j x^d(k) + F^{j-1} D^d F N \Delta u(k) + F^{j-2} D^d F N \Delta u(k+1) + \cdots + D^d F N \Delta u(k+m-1) \quad (14)$$

By applying Eq. 13, for the time steps inside the control horizon, we obtain the following expression

$$\bar{x}^s = \bar{I} x^s(k) + D_m^0 \Delta u_k \quad (15)$$

where

$$\bar{x}^s \triangleq \begin{bmatrix} x^s(k+1) \\ \vdots \\ x^s(k+m) \end{bmatrix}, \qquad \bar{x}^s \in \mathbb{R}^{mny} \qquad \bar{I} \triangleq \begin{bmatrix} I_{ny} \\ \vdots \\ I_{ny} \end{bmatrix}, \qquad \bar{I} \in \mathbb{R}^{mny\times ny};$$

$$D_m^0 \triangleq \begin{bmatrix} D^0 & & 0 \\ \vdots & \ddots & \\ D^0 & \cdots & D^0 \end{bmatrix}, \qquad D_m^0 \in \mathbb{R}^{m.ny\times m.nu}$$

$$\Delta u_k \triangleq \begin{bmatrix} \Delta u(k) \\ \vdots \\ \Delta u(k+m-1) \end{bmatrix} \qquad \Delta u_k \in \mathbb{R}^{m.nu}$$

For the same time steps, Eq. 14 produces

$$\begin{bmatrix} x^d(k+1) \\ x^d(k+2) \\ \vdots \\ x^d(k+m) \end{bmatrix} = \begin{bmatrix} F \\ F^2 \\ \vdots \\ F^m \end{bmatrix} x^d(k) + \begin{bmatrix} I & 0 & \cdots & 0 \\ F & I & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ F^{m-1} & F^{m-2} & \cdots & I \end{bmatrix} \times \begin{bmatrix} D^dN & 0 & \cdots & 0 \\ 0 & D^dN & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D^dN \end{bmatrix} \Delta u_k$$

$$\bar{x}^d = F_x x^d(k) + F_u \Delta u_k \quad (16)$$

With the variables defined above, the control objective represented in Eq. 5 can be written as

$$V_k = [\Delta u_k^T \quad \delta_k^T] H \begin{bmatrix} \Delta u_k \\ \delta_k \end{bmatrix} + 2c_f^T [\Delta u_k \delta_k] + c \quad (17)$$

where

$$H \triangleq \begin{bmatrix} (D_m^0 + \Psi_1 F_u)^T Q_1 (D_m^0 + \Psi_1 F_u) + F_u^T Q_2 F_u + R_1 \\ -(D_m^0 + \Psi_1 F_u)^T Q_1 \bar{I} \\ -\bar{I}^T Q_1 (D_m^0 + \Psi_1 F_u) \; S + \bar{I}^T Q_1 \bar{I} + Q \end{bmatrix} \quad (18)$$

$$c_f \triangleq \begin{Bmatrix} (D_m^0 + \Psi_1 F_u)^T Q_1 [\bar{I} e^s(k) + \Psi_1 F_x x^d(k)] \\ + F_u^T Q_2 [F_x x^d(k)] \\ -\bar{I}^T Q_1 [\bar{I} e^s(k) + \Psi_1 F_x x^d(k)] - Q e(k) \end{Bmatrix} \quad (19)$$

$$c \triangleq e(k)^T Q e(k) + [\bar{I} e^s(k) + \Psi_1 F_x x^d(k)]^T Q_1 [\bar{I} e^s(k) + \Psi_1 F_x x^d(k)] + [F_x x^d(k)]^T Q_2 [F_x x^d(k)] \quad (20)$$

$$Q_1 = \mathrm{diag}[\overbrace{Q \quad \cdots \quad Q}^{m}] \qquad Q_2 = \mathrm{diag}[\overbrace{0 \quad \cdots \quad 0 \quad \bar{Q}}^{m}]$$

$$R_1 = \mathrm{diag}[\overbrace{R \quad \cdots \quad R}^{m}]$$

$$\Psi_1 = \mathrm{diag}[\overbrace{\Psi \quad \cdots \quad \Psi}^{m}]$$

$$e^s(k) = x^s(k) - r$$

By using Eq. 13, the constraint represented in Eq. 8 can be written as follows

$$e^s(k) + \tilde{D}^0 \Delta u_k - \delta_k = 0 \tag{21}$$

where $\tilde{D}^0 = [D^0 \cdots D^0]$.

Finally, the control optimization problem of the extended infinite-horizon MPC (IHMPC) can be formulated as

Problem P1

$$\min_{\Delta u_k, \delta_k} V_k \tag{22}$$

subject to Eq. 21 and

$$\Delta u(k+j) \in U \qquad j \geq 0 \tag{23}$$

$$\mathbb{U} = \left\{ \Delta u(k+j) \left| \begin{array}{l} -\Delta u^{\max} \leq \Delta u(k+j) \leq \Delta u^{\max} \\ \Delta u(k+j) = 0 \qquad j \geq m \\ u^{\min} \leq u(k-1) + \sum_{i=0}^{j} \Delta u(k+i) \leq u^{\max} \qquad j = 0, 1, \ldots, m-1 \end{array} \right. \right\}$$

The convergence of the extended infinite-horizon MPC resulting from Problem P1 can be summarized in the following theorem:

Theorem 1. *For a stable system with a nonsingular coefficient matrix D^0 and an appropriate number of degrees of freedom, there is a matrix S such that the control law produced by the solution of Problem P1 drives the system output to its reference.*

Proof. Suppose that $[\Delta u_k^* \quad \delta_k^*]$ corresponds to the optimal solution of Problem P1 at time step k. It is easy to show that $[\Delta \tilde{u}_k \quad \delta_k^*]$ is a feasible solution to Problem P1 at $k + 1$. Control sequence $\tilde{u}$ is defined as $\Delta \tilde{u}_k = [\Delta u^*(k+1)^T \cdots \Delta u^*(k+m-1)^T \quad 0]^T$. Let the corresponding value of the objective function be designated $\tilde{V}_{k+1}$. It is straightforward to show that

$$\tilde{V}_{k+1} = V_k^* - [e(k) - \delta_k^*]^T Q [e(k) - \delta_k^*] - x^d(k+m)^T F^T \Psi^T Q \Psi F x^d(k+m) - \Delta u^*(k)^T R \Delta u^*(k)$$

If one of the terms $[e(k) - \delta_k]$, $x^d(k+m)$, or $\Delta u(k)$ is not equal to zero, then $\tilde{V}_{k+1} < V_k^*$ and consequently $V_{k+1}^* < V_k^*$. If this situation is repeated in the subsequent time steps $k+1$, $k+2$, . . . , the cost will converge to zero or to $\delta_\infty^T S \delta_\infty$ (with $\delta_\infty \neq 0$). In the first case, the system output will converge to the reference as k tends to infinite and convergence is proved. In the second case, the system output will not converge to the reference as $y(k) \to r + \delta_k$ when $k \to \infty$. However, if matrix D_0 is of full rank, it is possible to find a weight matrix S, such that the converged state is not the optimal one and we can find another solution to Problem P1, such that it corresponds to a cost smaller than $\delta_\infty^T S \delta_\infty$. To demonstrate this assertion, suppose that when $k \to \bar{k}$ (large enough) the state tends to the equilibrium point defined by $x^s(\bar{k})$ and $x^d(\bar{k})$. Remember that at this equilibrium point we are assuming that $\Delta u_{\bar{k}} = 0$ and suppose also that

$$e^s(\bar{k}) = x^s(\bar{k}) - r = \delta_k \neq 0$$

Because the system is stable, the dynamic components of the state vector tend to zero or $x^s(\bar{k}) = 0$, and consequently, at time step $\bar{k}$ the control objective will be reduced to

$$V_k = \delta_k^T S \delta_k$$

Now, to simplify the proof, assume that $m = 1$ and we search for a $\Delta u(\bar{k})$ such that the corresponding value of the control objective is smaller than $V_{\bar{k}}$. Because the constraint represented in Eq. 21 has to be satisfied by $\Delta u(\bar{k})$, we have

$$e^s(\bar{k}) + D^0 \Delta u(\bar{k}) - \bar{\delta}_k = 0$$

$$\delta_k + D^0 \Delta u(\bar{k}) - \bar{\delta}_k = 0$$

Suppose also that $\Delta u(\bar{k})$ is not constrained by its max or min bounds and, given that the model coefficient matrix D^0 is nonsingular, we can compute a control action $\Delta u(\bar{k})$ such that $\bar{\delta}_{\bar{k}} = 0$, or $\Delta u(\bar{k}) = -(D^0)^{-1} \delta_{\bar{k}}$.

With these assumptions, the corresponding control objective becomes

$$\bar{V}_k = e(\bar{k}+1)^T Q e(\bar{k}+1) + x^d(\bar{k}+1)^T \bar{Q} x^d(\bar{k}+1) + \Delta u(\bar{k})^T R \Delta u(\bar{k}) \tag{24}$$

By substituting Eqs. 12, 13, and 14 into Eq. 24, this last equation can be simplified to

$$\bar{V}_k = \delta_k^T Z \delta_k$$

where

$$Z = [D^d F N (D^0)^{-1}]^T \bar{Q} [D^d F N (D^0)^{-1}] + [(D^0)^{-1}]^T R (D^0)^{-1}$$

Hence, the output will converge to the reference if S is such that

$$\bar{V}_{\bar{k}} < V_{\bar{k}}$$

or

$$Z < S \tag{25}$$

Therefore, if the weight of the slack variables is large enough to satisfy Eq. 25, the controller defined by Problem P1 will reduce the cost asymptotically to zero and convergence is proved.

To simplify the proof of stability, assume again that $m = 1$, and at sampling step k consider the value of the cost function corresponding to $\Delta u_k = 0$, which is a feasible solution to Problem P1

$$\tilde{V}_k = \sum_{j=0}^{\infty} [x^s(k+j) + \Psi x^d(k+j) - \tilde{\delta}_k - r]^T \times Q[x^s(k+j) + \Psi x^d(k+j) - \tilde{\delta}_k - r] + \tilde{\delta}_k^T S \tilde{\delta}_k$$

where from the equality constraint 21 we have $\tilde{\delta}_k = e^s(k)$, and considering the terminal state weight computed in Eq. 10, the above cost becomes

$$\tilde{V}_k = x^d(k)^T \bar{Q} x^d(k) + e^s(k)^T S e^s(k)$$

$$\tilde{V}_k = [e^s(k)^T \quad x^d(k)^T] \begin{bmatrix} S & 0 \\ 0 & \bar{Q} \end{bmatrix} \begin{bmatrix} e^s(k) \\ x^d(k) \end{bmatrix}$$

Assume now that Problem P1 is solved at successive time steps $k, k+1, \ldots,$ and we reach time step $k+n$. At this time, a feasible solution is $\Delta u_{k+n} = 0$ and $\delta_{k+n} = \delta^*_{k+n-1}$, where δ^*_{k+n-1} corresponds to the optimal solution of the previous step and can be obtained from the equality constraint represented in Eq. 21, which in this case becomes

$$e^s(k+n) - \delta^*_{k+n-1} = 0$$

This solution corresponds to the following cost

$$\tilde{V}_{k+n} = x^d(k+n)^T \bar{Q} x^d(k=n) + e^s(k+n)^T S e^s(k+n)$$

$$\tilde{V}_{k+n} = [e^s(k+n)^T \quad x^d(k+n)^T] \begin{bmatrix} S & 0 \\ 0 & \bar{Q} \end{bmatrix} \begin{bmatrix} e^s(k+n) \\ x^d(k+n) \end{bmatrix}$$

It is clear that for the undisturbed system $\tilde{V}_{k+n} \le V^*_{k+n-1} \le \tilde{V}_k$, and consequently

$$[e^s(k+n)^T \quad x^d(k+n)^T] \begin{bmatrix} S & 0 \\ 0 & \bar{Q} \end{bmatrix} \begin{bmatrix} e^s(k+n) \\ x^d(k+n) \end{bmatrix} \le [e^s(k)^T \quad x^d(k)^T] \begin{bmatrix} S & 0 \\ 0 & \bar{Q} \end{bmatrix} \begin{bmatrix} e^s(k) \\ x^d(k) \end{bmatrix}$$

Because S and $\bar{Q}$ are positive definite, it follows that

$$\left\| \begin{bmatrix} e^s(k+n) \\ x^d(k+n) \end{bmatrix} \right\| \le \alpha \left\| \begin{bmatrix} e^s(k) \\ x^d(k) \end{bmatrix} \right\|$$

where $\|[\cdot]\|$ stands for the Euclidean norm of $[\cdot]$ and

$$\alpha = [\lambda_{\max}(G)/\lambda_{\min}(G)]^{1/2} \qquad G = \begin{bmatrix} S & 0 \\ 0 & \bar{Q} \end{bmatrix}$$

If the initial state is restricted to

$$\left\| \begin{bmatrix} e^s(k) \\ x^d(k) \end{bmatrix} \right\| \le \rho$$

then, at any subsequent time instant the state will be restricted to

$$\left\| \begin{bmatrix} e^s(k+n) \\ x^d(k+n) \end{bmatrix} \right\| \le \alpha\rho$$

Remark 1

In the above theorem, by an appropriate number of degrees of freedom, we mean that the inputs, which are not saturated, are such that the system can be stabilized at the desired output reference. If the system does not have the required degrees of freedom, the closed-loop system will tend to stabilize at an equilibrium point that corresponds to a cost $V_k = \delta_\infty^T S \delta_\infty \neq 0$.

Remark 2

When solving Problem P1, we assume that Eq. 10 has been already solved to obtain the terminal state penalty $\bar{Q}$. The solution of Eq. 10 can be simplified considering that, the output weight matrix Q is usually a diagonal matrix $Q = \text{diag}([q_1 \quad q_2 \cdots q_{ny}])$, where q_i is the weight of output y_i. Also, for systems with distinct poles as assumed here, matrix F is also diagonal

$$F = \text{diag}([F_1 \quad F_2 \quad \cdots \quad F_{ny}])$$

$$F_i = \text{diag}([f_1 \quad f_2 \quad \cdots \quad f_{nr}]_i), \qquad [f_p]_i = e^{r_{i,j,l}T},$$
$$p = (j-1)na + l, \qquad nr = nuna \qquad i = 1, \ldots, ny;$$
$$j = 1, \ldots, nu; \, l = 1, \ldots, na.$$

Based on the these observations, we conclude that $\bar{Q}$ can be a block diagonal matrix: $\bar{Q} = \text{diag}([\bar{Q}_1 \quad \tilde{Q}_2 \cdots \bar{Q}_{ny}[)$, where $\bar{Q}_i \in \mathbb{C}^{nr \times nr}$ and, consequently, Eq. 10 can be separated into ny independent equations as follows

$$\bar{Q}_i - F_i^T \bar{Q}_i F_i = F_i^T \Psi^T Q \Psi F_i \qquad i = 1, \ldots, ny \tag{26}$$

where

$$\bar{Q}_i = \begin{bmatrix} \bar{q}_{1,1} & \cdots & \bar{q}_{1,np} \\ \vdots & \ddots & \vdots \\ \bar{q}_{np,1} & \cdots & \bar{q}_{np,np} \end{bmatrix}_i$$

Now, define

$$\bar{G}_i \triangleq F_i^T \Psi^T Q \Psi F_i$$

$$\bar{G}_i = \begin{bmatrix} \bar{g}_{1,1} & \cdots & \bar{g}_{1,np} \\ \vdots & \ddots & \vdots \\ \bar{g}_{np,1} & \cdots & \bar{g}_{np,np} \end{bmatrix}_i$$

The solution of Eq. 26 can be summarized in the following expression

$$[\bar{q}_{j,l}]_i = \frac{[\bar{g}_{j,l}]_i}{(1 - [f_j]_i[f_l]_i)} \qquad i = 1, \ldots, ny; \quad j = 1, \ldots, np; \; l = 1, \ldots, np \tag{27}$$

Thus, for systems with distinct poles, with the adopted model formulation, the solution of the Lyapunov equation is straightforward and produces a negligible increase in the computer demand. Thus, whenever there is a change in the controller configuration in terms of controlled outputs and manipulated inputs, Eq. 10 is used to compute the terminal weight corresponding to the actual control configuration.

In the next section we apply the extended infinite-horizon MPC presented above to two controllers of the MPC literature that were shown to enforce robust stability when operating as regulators. Recall that in the regulator problem, it is assumed that the system equilibrium point is known and the model inserted in the MPC deals with input and output deviations from the equilibrium point. For most chemical processes, the actual equilibrium point is not completely known because of unmeasured disturbances and system nonlinearity.

Extended Robust MPC

To characterize model uncertainty, we assume that matrices A and B of the model represented in Eq. 1 are not exactly known, but the model is known to lie within a set Ω of stable models with the same dimensions. To simplify notation, we designate $\theta \triangleq (A, B)$ and any individual plant corresponds to a particular θ. The nominal or most probable plant is represented by θ_N and the true plant is represented by θ_T. To make explicit the dependency of the control objective represented in Eq. 17 on model parameters, it is convenient to express the objective as follows

$$V_k[\Delta u_k, \delta_k(\theta), \theta] = [\Delta u_k^T \quad \delta_k^T(\theta)]H(\theta)\begin{bmatrix}\Delta u_k \\ \delta_k(\theta)\end{bmatrix} + 2c_f^T(\theta)\begin{bmatrix}\Delta u_k \\ \delta_k(\theta)\end{bmatrix} + c(\theta) \tag{28}$$

To the incremental system, we now extend one of the most popular approaches of the robust MPC literature. This controller is based on the inclusion of an explicit constraint on the control objective.

Cost-constraining MPC

The extended infinite-horizon MPC introduced in the previous section allows the generalization of the robust linear quadratic regulator developed by Badgwell (1997) for the multiplant system and Ralhan and Badgwell (2000) for the system with continuous uncertainty. The extended cost-constraining robust MPC is obtained as the solution of the following optimization problem.

Problem P2

$$\min_{\Delta u_k, \delta_k(\theta)} \; V_k[\Delta u_k, \delta_k(\theta_N), \theta_N] \tag{29}$$

subject to Eq. 23 and

$$V_k[\Delta u_k, \delta_k(\theta), \theta] \le V_k[\Delta \tilde{u}_k, \tilde{\delta}_k(\theta), \theta] \qquad \theta \in \Omega \tag{30}$$

$$e^s(k) + \tilde{D}^0(\theta)\Delta u - \delta_k(\theta) = 0 \qquad \theta \in \Omega \tag{31}$$

where $\Delta \tilde{u}_k$ is the optimal control sequence obtained at time step $(k - 1)$ and translated to time k, and $\tilde{\delta}_k(\theta)$ is such that

$$e^s(k) + \tilde{D}^0(\theta)\Delta \tilde{u}_k - \tilde{\delta}_k(\theta) = 0 \qquad \theta \in \Omega \tag{32}$$

The introduction of $\tilde{\delta}_k(\theta)$ is necessary to accommodate the feedback from the state of the true plant to other models lying within Ω. It is clear that, for the undisturbed true system, $\tilde{\delta}_k(\theta_T) = \delta^*_{k-1}(\theta_T)$.

The following theorem shows that the control algorithm produced by the solution of Problem P2 provides convergence of the true system output to the reference value.

Theorem 2. *Consider a stable system whose true model is unknown but it is known to lie within the set Ω. Assume that in the control objective V_k and weights Q, R, and S are such that Inequality 25 holds true for all models lying within Ω. Assume also that Problem P2 is solved at time steps k, k + 1, k + 2, ⋯ and the system inputs do not become saturated. Then, the control law obtained as a solution to Problem P2 drives the true system output to the reference value.*

Proof. Suppose that at time k, Problem P2 is solved and the resulting optimal solution is represented by $[\Delta u_k^*, \delta_k^*(\theta)]$, $\theta \in \Omega$. *For the true plant, the corresponding cost is*

$$V_k[\Delta u_k^*, \delta_k^*(\theta_T), \theta_T] = \sum_{j=0}^{\infty} [e(k+j) - \delta^*(\theta_T)]^T Q[e(k+j) - \delta^*(\theta_T)] + \sum_{j=0}^{m-1} \Delta u^*(k+j)^T R \Delta u^*(k+j) + \delta_k^*(\theta_T)^T S \delta_k^*(\theta_T)$$

Assume that we inject the first control action $\Delta u^*(k)$ into the true plant and move time to $k + 1$. At this time step, consider the value of the control objective for $[\Delta \tilde{u}_k, \tilde{\delta}_k(\theta_T), \theta_T]$

$$\begin{aligned} V_{k+1}[\Delta \acute{u}_k, \tilde{\delta}_k(\theta_T), \theta_T] &= V_k[\Delta u_k^*, \delta_k^*(\theta_T), \theta_T] \\ &- [e(k) - \delta_k^*(\theta_T)]^T Q[e(k) - \delta_k^*(\theta_T)] - \Delta u^*(k)^T R \Delta u^*(k) \\ &\le V_k[\Delta u_k^*, \delta_k^*(\theta_T), \theta_T] \end{aligned}$$

Thus, if problem P2 is feasible at $k + 1$ and the cost for the true plant will satisfy the relation

$$V_{k+1}[\Delta u_{k+1}^*, \delta_{k+1}^*(\theta_T), \theta_T] \le V_k[\Delta u_k^*, \delta_k^*(\theta_T), \theta_T]$$

Consequently, $V_k[\Delta u_k, \delta_k(\theta_T), \theta_T]$, which is positive and bounded below by zero, is also nonincreasing. Thus, assuming that the inputs are not saturated and Eq. 25 holds true for all the

Table 1. Tuning Parameters of the Cost-Constraining Robust MPC

T	m	Q	R	S	Δu_{max}	Δu_{min}	u_{max}	u_{min}
2	3	diag(1, 1)	diag(10^{-2}, 10^{-2})	diag(10^2, 10^2)	0.25	−0.25	1.5	−1.5

models lying within Ω, the control objective will converge to zero for the true plant. Similarly to the nominal case treated in Theorem 1, for a square system, if one of the inputs becomes saturated, the resulting cost will still be bounded but it will not converge to zero because there will not be enough degrees of freedom to reduce the cost to zero. As a consequence, the system outputs will converge to an equilibrium point, which does not correspond to the reference.

Stability of the closed-loop system, with the control law produced by the solution of Problem P2, can be proved following the same steps of the proof of stability in Theorem 1. Here, we consider the cost associated with the true plant and observe that constraint 30 prevents this cost from increasing.

Remark 3

Problem P2 proposed above is of infinite dimension for any compact set Ω. This is so because for each θ there is a vector$\delta_k(\theta)$, which becomes part of the set of decision variables of the control optimization problem. However, this problem can be simplified because $\delta_k(\theta)$ is not constrained, and for each $\delta_k(\theta)$ there is an equality constraint represented in Eq. 31. Therefore, the slacks can be eliminated from the optimization problem. After the substitution of $\delta_k(\theta)$, the left-hand side of Eq. 32 can be written as follows

$$\tilde{V}_k(\Delta u_k,\ \theta) = \Delta u_k^T \tilde{H}(\theta)\Delta u_k + 2\tilde{c}_f^T(\theta)\Delta u_k + \tilde{c}(\theta) \quad (33)$$

where

$$H(\theta) = \begin{bmatrix} H_{11}(\theta) & H_{12}(\theta) \\ H_{21}(\theta) & H_{22} \end{bmatrix} \qquad c_f^T(\theta) = [c_{f,1}^T(\theta) \quad c_{f,2}^T(\theta)]$$

$$\tilde{H}(\theta) = H_{11}(\theta) + H_{12}(\theta)\tilde{D}^0(\theta) + \tilde{D}^0(\theta)^T H_{21}(\theta) + \tilde{D}^0(\theta)^T H_{22}(\theta)\tilde{D}^0(\theta)$$

$$\tilde{c}_f^T(\theta) = e^s(k)^T[H_{21}(\theta) + H_{22}(\theta)\tilde{D}^0(\theta)] + c_{f1}^T(\theta) + c_{f2}^T(\theta)\tilde{D}^0(\theta)$$

$$\tilde{c}(\theta) = e^s(k)^T H_{22} e^s(k) + 2c_{f2}^T(\theta)e^s(k) + c(\theta)$$

$H(\theta)$, cf, and c are defined in Eqs. 18, 19, and 20, respectively.

With this reduced expression for the cost, the control law for the cost-constraining robust MPC can be obtained as the solution of the following optimization problem

Problem P3

$$\min_{\Delta u_k} \tilde{V}_k(\Delta u_k,\ \theta_N) \quad (34)$$

subject to Eq. 23 and

$$\tilde{V}_k(\Delta u_k,\ \theta) \le V_k(\Delta \tilde{u}_k,\ \theta) \qquad \theta \in \Omega \quad (35)$$

We now illustrate the application of the generalized robust cost-constraining MPC with the simulation of a classical system of the process control literature.

Example 1

Ralhan and Badgwell (2000) studied the robust control of a high purity distillation column and compared the performance of several regulators, which are robust to model uncertainty. Here, we simulate this system with the controller defined by Problem P3. Given below we have the model that relates the outputs: distillate composition (y_1) and bottom stream composition (y_2), to the inputs: reflux flow rate (u_1) and reboiler vapor flow rate

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{22.98}\begin{bmatrix} 0.7868(1+\gamma_1) & -0.6147(1+\gamma_2) \\ 0.8098(1+\gamma_1) & -0.982(1+\gamma_2) \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

In this model, γ_1 and γ_2 are parameters that correspond to model uncertainty. It is assumed that $-0.4 \le \gamma_1$ and $\gamma_2 \le 0.4$. Ralhan and Badgwell (2000) compare several methods to solve the semi infinite optimization problem of the cost-constraining regulator. Any of those methods can be applied to the solution of Problem P3. Here, we adopt the simplest method in which set Ω is approximated by a discrete set of models. In the simulation performed here, set Ω is approximated by the following set

$$\Omega' = [(0,0) \quad (-0.4, -0.4) \quad (-0.4, 0.4) \quad (0.4, -0.4) \quad (0.4, 0.4)]$$

It is assumed that the nominal model corresponds to $(\gamma_1, \gamma_2) = (0, 0)$. In the simulation of Case 1, the nominal model represents the true plant. In the simulation Cases 2 and 3, the true plant is represented by $(\gamma_1, \gamma_2) = (-0.4, -0.4)$ and $(\gamma_1, \gamma_2) = (-0.4, 0.4)$, respectively. In these three cases, the following scenario is simulated: at time 0, the system starts at the origin; the output reference is changed to $(y_1^{set}, y_2^{set} = (0.1, 0)$, at sample step 50, an unmeasured disturbance is introduced, corresponding to $(\Delta u_1, \Delta u_2) = (0.5, 0.5)$. The tuning parameters of the controller are shown in Table 1.

Figure 1 shows the system responses with the robust controller for the three cases simulated. We observe that, as noted by Ralhan and Badgwell (2000), a better performance is obtained when the plant model is the nominal model. We may also note that the introduction of a disturbance that moves the steady state of the input to a different value does not affect the convergence of the controller, and neither does it cause any infeasibility to the control optimization problem.

Another simulation was performed to verify the effect of the

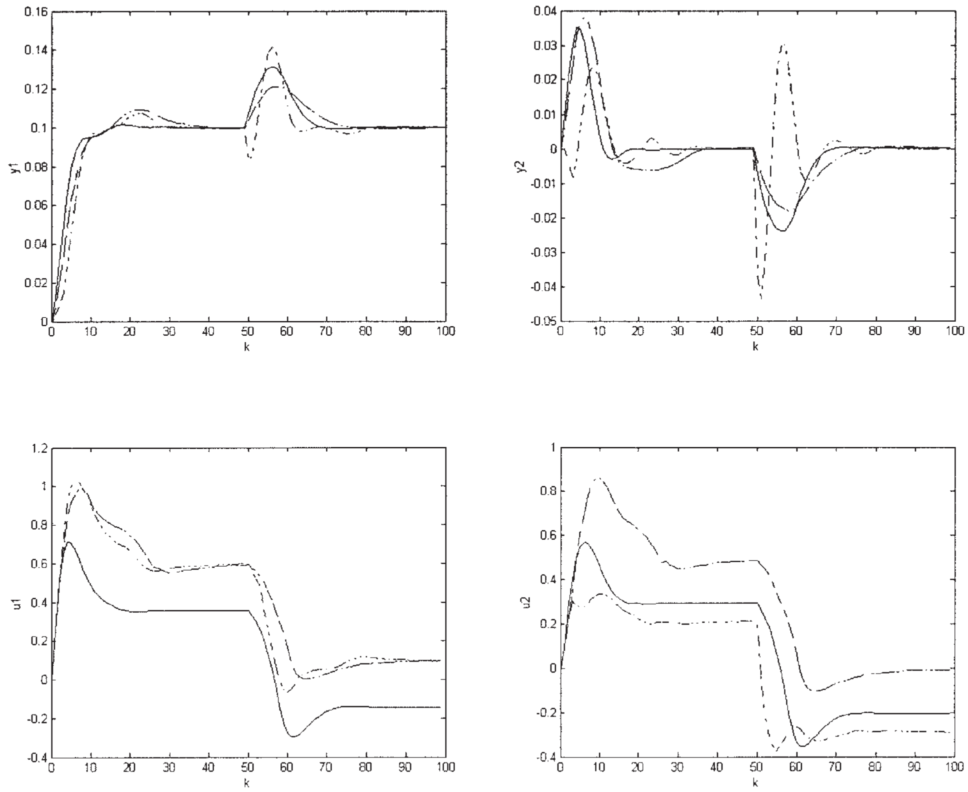

Figure 1. Robust control of the distillation system.

Case 1 (—), Case 2 (– – –), Case 3 (– · —).

saturation of the input on the convergence of the controller, a case considered similar to Case 2 with the same plant but without the unmeasured disturbance. The max bound on input u_1 was reduced to 0.5, which is low enough such that u_1 becomes saturated. Figure 2 shows the responses of the system for this case, and we can observe that the system outputs still converge to a steady state but, as a consequence of the input saturation, the output does not converge to its reference; nor does y_1 tend to its reference, which is 0.1; nor does y_2 tend to zero. For the same case, Figure 3 shows the cost function (V_k) for the true plant. This cost is nonincreasing indicating convergence but, because of the input saturation, the cost cannot be reduced to zero.

Min–max MPC

In this section, we apply the generalized infinite-horizon MPC presented in this work to the robust regulator developed by Lee and Yu (1997) and referred to by these authors as open-loop worst-case optimal feedback control (OLWOFC). The main feature of this controller is that, although in the control optimization problem it is assumed that the input sequence is in open loop, for the output prediction, it takes into account the state feedback of all models lying within Ω. To adapt OLWOFC to the model notation used in this work, we consider the following vector of model parameters

$$\Theta_k \triangleq [\theta_k \quad \theta_{k+1} \quad \cdots \quad \theta_{k+m}]^T$$

where $\theta_{k+i} \in \Omega$ $(i = 0, 1, 2, \ldots, m)$.

For a particular combination of model parameters, we define the control objective as follows

$$V_k[\Delta u_k, \delta_k(\Theta_k), \Theta_k] \triangleq [u_k^T \quad \delta_k^T(\Theta_k)]H(\Theta_k)\begin{bmatrix} \Delta u_k \\ \delta_k(\Theta_k) \end{bmatrix} + 2c_f^T(\Theta_k)\begin{bmatrix} \Delta u_k \\ \delta_k(\Theta_k) \end{bmatrix} + c(\Theta_k)$$

where $H(\Theta_k)$, $c_f(\Theta_k)$, and $c(\Theta_k)$ can be computed by Eqs. 18, 19, and 20, respectively, and considering the following definitions

$$D_m^0(\Theta_k) \triangleq \begin{bmatrix} D^0(\theta_k) & & & 0 \\ D^0(\theta_k) & D^0(\theta_{k+1}) & & \\ \vdots & \vdots & \ddots & \\ D^0(\theta_k) & D^0(\theta_{k+1}) & \cdots & D^0(\theta_{k+m-1}) \end{bmatrix}$$

$$F_x(\Theta_k) \triangleq \begin{bmatrix} F_1(\Theta_k) \\ F_2(\Theta_k) \\ \vdots \\ F_{m-1}(\Theta_k) \end{bmatrix}$$

$$F_i(\Theta_k) \triangleq F(\theta_k)F(\theta_{k+1}) \cdots F(\theta_{k+i-1})$$

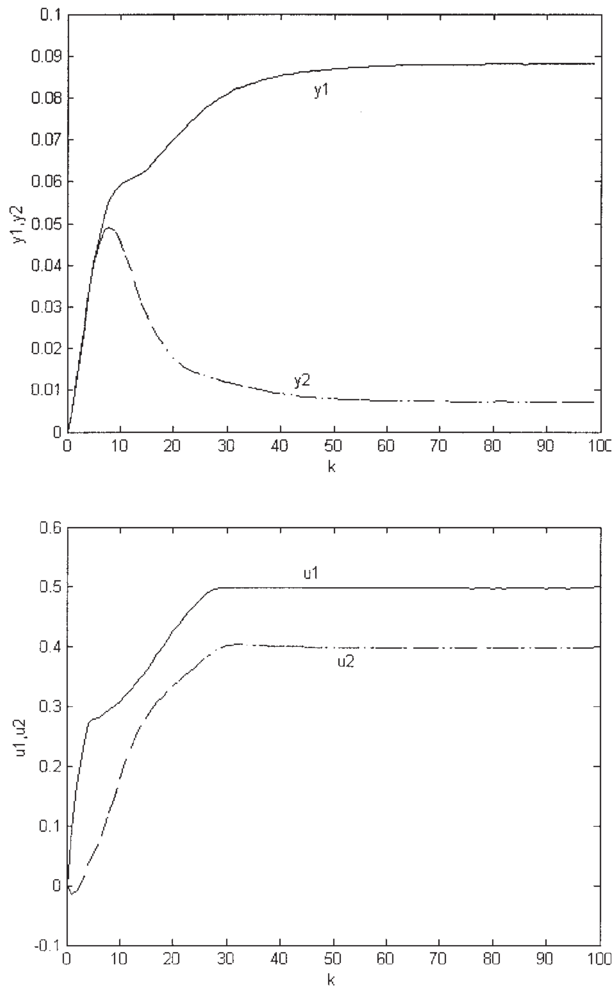

Figure 2. Responses of the distillation system with input saturation.

$$F_u(\Theta_k) = \begin{bmatrix} F_1(\Theta_k) & 0 & \cdots & 0 \\ F_2(\Theta_k) & F_1(\Theta_k) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ F_{m-1}(\Theta_k) & F_{m-2}(\Theta_k) & \cdots & F_1(\Theta_k) \end{bmatrix} \times \begin{bmatrix} D^d(\theta_1)N & 0 & 0 & 0 \\ 0 & D^d(\theta_2)N & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & D^d(\theta_{m-1})N \end{bmatrix}$$

In the above objective, the terminal cost is computed using Eq. 10 as follows

$$\bar{Q}(\Theta_k) - F(\theta_{k+m})^T \bar{Q}(\Theta_k) F(\theta_{k+m}) = F(\theta_{k+m})^T \Psi^T Q \Psi F(\theta_{k+m})$$

Finally, in the generalized OLWOFC, the input sequence is determined by solving the following optimization problem:

Problem P4

$$\min_{\Delta u_k, \delta_k(\Theta_k)} \max_{\Theta_k \in \Omega^m} V_k[\Delta u_k, \delta_k(\Theta_k), \Theta_k] \qquad (36)$$

subject to Eq. 23 and

$$e^s(k) + \tilde{D}^0(\Theta_k)\Delta u_k - \delta_k(\Theta_k) = 0$$
$$\Theta_k \in \Omega^m \triangleq \underbrace{\Omega \times \cdots \times \Omega}_{m} \qquad (37)$$

The optimization problem defined above is also of infinite dimension, given that, in the general case, there are infinite realizations of the parameter vector Θ_k and for each of these realizations there is a vector of decision variables $\delta_k(\Theta_k)$. This problem is also nonlinear and nonconvex, although if a solution can be found, convergence for the control law resulting from this solution is stated through the following theorem:

Theorem 3. *Consider a stable system whose true model is unknown but it is known to lie within the set* Ω. *Assume that in the control objective* V_k *and weights Q, R, and S are such that Inequality* 25 *holds true for all the models in* Ω. *Then, the control law resulting from the solution of Problem P4 drives the system output to the reference value.*

Proof. The proof can be developed following the same steps of Lee and Cooley (2000). Suppose that at time step k, Problem P4 is solved and the resulting optimal solution is represented by

$$\Phi_k \triangleq \min_{\Delta u_k} \max_{\Theta_k \in \Omega^m} V_k(\Delta u_k, \delta_k(\Theta_k), \Theta_k)$$

which corresponds to the decision variables $[\Delta u_k^*, \delta_k^*(\Theta_k)]$ for any $\Theta_k \in \Omega^m$ Then, it is clear that

$$\Phi_k \geq \min_{\Delta u_k} \max_{\Theta_k \in \Omega^m} V_k[\Delta u_k, \delta_k(\Theta_k), \Theta_k]$$

with $\theta_k = \theta_T$ and $\Delta u(k) = \Delta u^*(k)$.

$$\Phi_k \geq \min_{\Delta u_{k+1}} \max_{\Theta_{k+1} \in \Omega^m} V_{k+1}[\Delta u_{k+1}, \delta_{k+1}(\Theta_{k+1}), \Theta_{k+1}] + [e(k+1) - \delta_{k+1}(\Theta_{k+1})]^T Q [e(k+1) - \delta_{k+1}(\Theta_{k+1})] + \Delta u^*(k)^T R \Delta u^*(k)$$

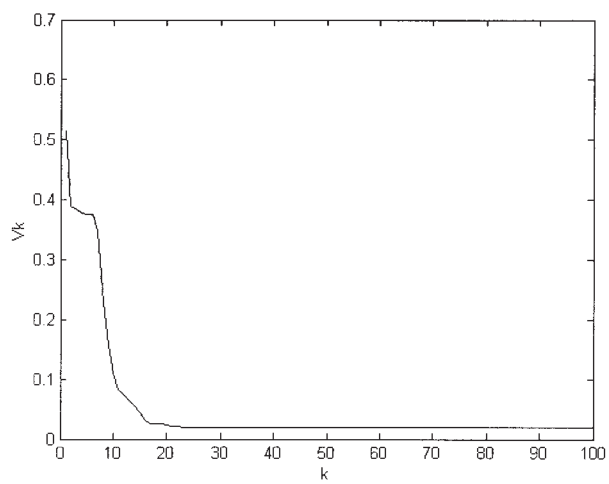

Figure 3. Cost for the true distillation system with input saturation.

Table 2. Models of the Debutanizer Column

$$G_1(s) = \begin{bmatrix} \dfrac{-0.2623}{60s^2 + 59.2s + 1} & \dfrac{0.1368}{1164s^2 + 99.7s + 1} \\ \dfrac{0.1242}{218.7s^2 + 16.2s + 1} & \dfrac{-0.1351}{70s^2 + 20s + 1} \end{bmatrix} \quad G_2(s) = \begin{bmatrix} \dfrac{-0.3544}{218.6s^2 + 50.1s + 1} & \dfrac{0.2044}{1150s^2 + 93.86s + 1} \\ \dfrac{0.0685}{100.2s^2 + 11.32s + 1} & \dfrac{-0.1256}{20s^2 + 15s + 1} \end{bmatrix} \quad G_3(s) = \begin{bmatrix} \dfrac{-0.279}{59.77s^2 + 99.61s + 1} & \dfrac{0.050}{499.8s^2 + 73.77s + 1} \\ \dfrac{0.1950}{220.1s^2 + 18.93s + 1} & \dfrac{-0.1722}{29.74s^2 + 20.71s + 1} \end{bmatrix}$$

with $\Delta u(k + m) = 0$.

$$\Phi_k \geq \min_{\Delta u_{k+1}} \max_{\Theta_{k+1}\in\Omega^m} V_{k+1}[\Delta u_{k+1}, \delta_{k+1}(\Theta_{k+1}), \Theta_{k+1}] + [e(k+1) - \delta_{k+1}(\Theta_{k+1})]^T Q[e(k+1) - \delta_{k+1}(\Theta_{k+1})] + \Delta u^*(k)^T R \Delta u^*(k)$$

$$\Phi_{k+1} \geq \Phi_{k+1} + [e(k+1) - \delta_{k+1}(\Theta_{k+1})]^T Q[e(k+1) - \delta_{k+1}(\Theta_{k+1})] + \Delta u^*(k)^T R \Delta u^*(k)$$

Thus, the cost is nonincreasing and, if the inputs are not saturated and Inequality 25 is satisfied by all the models lying within Ω, the system output will converge to the reference, the slacks will converge to zero, and Δu_k will also converge to zero as k{rarrow} ∞.

Remark 4

Analogously to the robust cost-constraining MPC, the min–max robust control problem can be simplified by substituting Eq. 37 into the control objective to eliminate the slacks $\delta_k(\Theta_k)$ from Problem P4. The result is the following optimization problem:

Problem P5

$$\min_{\Delta u_k} \max_{\Theta_k\in\Omega^m} \bar{V}_k(\Delta u_k, \Theta_k) \tag{38}$$

subject to Eq. 23, where

$$\bar{V}_k(\Delta u_k, \Theta_k) = \Delta u_k^T \bar{H}(\Theta_k)\Delta u_k + 2\bar{c}_f^T(\Theta_k)\Delta u_k + \bar{c}(\Theta_k) \tag{39}$$

$$H(\Theta_k) = \begin{bmatrix} H_{11}(\Theta_k) & H_{12}(\Theta_k) \\ H_{21}(\Theta_k) & H_{22} \end{bmatrix} \quad c_f^T(\Theta_k) = [c_{f,1}^T(\Theta_k) \quad c_{f,2}^T(\Theta_k)]$$

$$\bar{H}(\Theta_k) = H_{11}(\Theta_k) + H_{12}(\Theta_k)\tilde{D}^0(\Theta_k) + \tilde{D}^0(\Theta_k)^T H_{21}(\Theta_k) + \tilde{D}^0(\Theta_k)^T H_{22}(\Theta_k)\tilde{D}^0(\Theta_k)$$

$$\tilde{c}_f^T(\Theta_k) = e^s(k)^T[H_{21}(\Theta_k) + H_{22}(\Theta_k)\tilde{D}^0(\Theta_k)] + c_{f,1}^T(\Theta_k) + c_{f,2}^T(\Theta_k)\tilde{D}^0(\Theta_k)$$

$$\tilde{c}(\Theta_k) = e^s(k)^T H_{22} e^s(k) + 2c_{f,2}^T(\Theta_k)e^s(k) + c(\Theta_k)$$

We now exemplify the application of the extended robust min–max MPC, based on Problem P5 above, to a practical system. Here we simulate the robust control of an industrial debutanizer that operates in a broad range of process conditions and, depending on the operating point, it can be represented by a different linear model (Rodrigues and Odloak, 2003).

Example 2

The simulated process has two controlled outputs: the pentane concentration in the top stream (y_1) and the vapor pressure of gasoline (y_2), which is produced in the bottom stream. The manipulated inputs are the top reflux flow (u_1) and the reboiler heat load (u_2). In the control of the debutanizer system, model set Ω was discretized and represented by three models. The models considered here are shown in Table 2.

Figure 4 shows the responses of the extended robust min–max MPC for the case where the output references were moved from (0, 0) to (1, 0) at time step $k = 0$. The controller is based on the three models listed in Table 2 and the following tuning parameters were used: $T = 3$, $m = 1$, $Q = \text{diag}(1, 1)$, $R = \text{diag}(0.05, 0.05)$, and $S = \text{diag}(25, 25)$. Three cases are considered depending on the true plant being represented by model $G_1(s)$, $G_2(s)$, or $G_3(s)$, respectively. At time step $k = 200$, an unmeasured input disturbance is introduced that moves the input steady state to an unknown value. We observe that, in all cases, the system outputs converge to the desired references. A small control horizon was used because this method usually needs a high computer demand. The performance of the controller can be considered acceptable from the perspective of practicality. Figure 5 shows the worst value of control objective for the same simulated cases. We observe that the input disturbance, introduced into the system at $k = 200$, forces V_k to increase before it finally converges to zero. In the simulated cases, it was not considered the input saturation. If this happens, the behavior of the min–max controller would be similar to the behavior of the cost-constraining controller illustrated in Example 1.

Remark 5

In Problem P4, it is assumed that any plant lying within Ω can be the true plant whose state is measured and produce the feedback to the MPC algorithm. To take this effect into account in the synthesis of the robust control law, all possible model combinations are considered along the control horizon. This is equivalent to considering that the true plant model can change within Ω at every sample step. This is a rather unrealistic scenario, particularly for chemical processes where the time scale for changes in the process model is expected to be much larger than the usual range of control horizons of MPC. As pointed out by Lee and Cooley (2000), when the model parameters can be considered constant, the min–max controller produces a conservative control law. To overcome this problem and still preserve robustness, they suggest including, for instance, the cost-constraining constraint into the min–max controller with constant model parameters. The corresponding version of the generalized robust controller is obtained from the solution of the following problem:

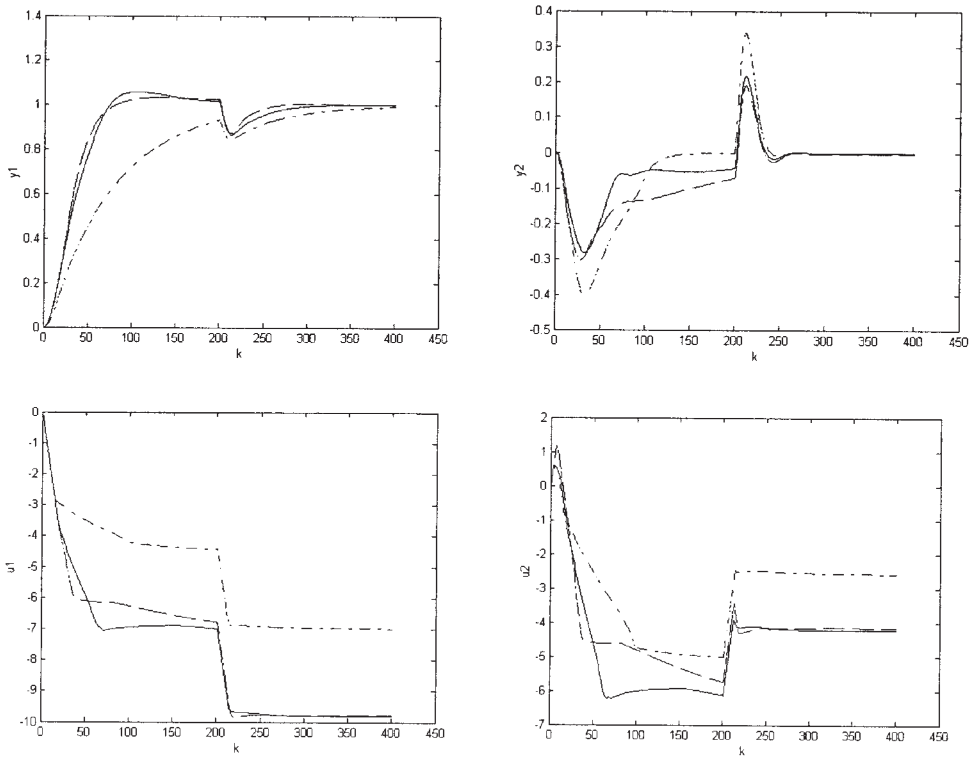

Figure 4. Min–max robust control of the debutanizer column.

Case 1 (—), Case 2 (– – –), Case 3 (– · —).

Problem P6

$$\min_{\Delta u_k, \delta_k(\theta)} \max_{\theta \in \Omega} V_k[\Delta u_k, \delta_k(\theta), \theta] \tag{40}$$

subject to Eqs. 23, 30, and 31.

The order of the above problem can be reduced as in Problems P2 and P4, by substituting $\delta_k(\theta)$ obtained from Eq. 31 into the control objective and other constraints.

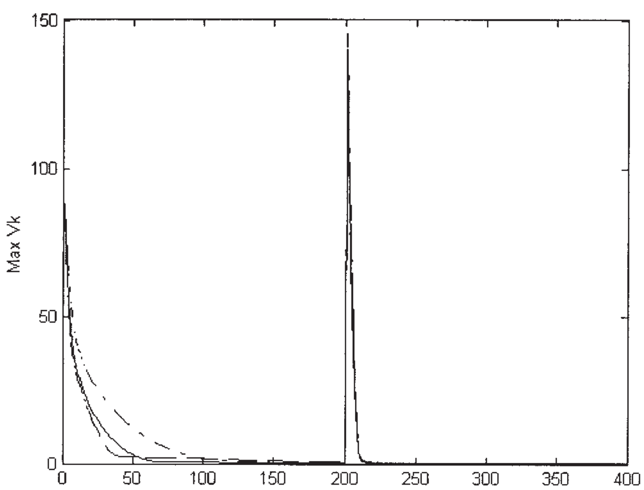

Figure 5. Maximum cost for the min–max robust MPC of the debutanizer column.

Case 1 (—), Case 2 (– – –), Case 3 (– · —).

Conclusions

We have presented herein a method to generalize robust MPC controllers that are based on the infinite-horizon model predictive controller (IHMPC). These controllers were presented in the control MPC literature by other authors for the case in which the control loop works as a regulator. In that case, it was assumed that the desired equilibrium point of the system is at the origin. The procedure is detailed for two well-known controllers: the cost-constraining robust MPC and the min–max MPC. The method presented here opens the possibility for the application of these controllers to practical cases where the control problem cannot be reduced to the regulator problem because of unknown disturbances or model nonlinearities.

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Manuscript received Feb. 27, 2003, and revision received Nov. 25, 2003.